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SLOPE-DEFLECTION METHOD FOR ELASTIC-PLASTIC MULTISTORY FRAMES†

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Abstract—Slope-Deflection Method for elastic rigid frames is generalized for elastic-plastic analysis of rigid frames of work-hardening materials. The plastic strain is here treated as an additional set of externally applied moments. The end moments of the component members of the frame are obtained from the solution of a set of linear algebraic equations. No iteration is required. Numerical results of the analysis of a portal frame and a two-story plane frame subject to side loads beyond the elastic range are shown.

INTRODUCTION

Ductile structural materials can withstand much strain beyond their elastic range. For rigid frames, this strain induces a redistribution of stresses which often results in a considerable additional capacity for carrying loads. Hence elastic analysis is unduly conservative. To determine the additional load-carrying capacity, the effect of plastic strain has to be considered. Methods of limit analysis, developed by many distinguished investigators[1–8], are applicable mainly to structures of perfectly plastic materials and not applicable to structures of strain-hardening materials. Metals (except low carbon steels) generally have strain hardening. To find the load carrying capacity of rigid frames of such materials, elastic-plastic analysis is needed.

A rigid frame may have component members subject to both bending and compression. These members behave like beam-columns. If the axial loads is small as compared to the critical column load of the member, the column effect may be neglected. In 1960, Ang[9] showed an iteration method to analyze a elastic-plastic portal frame subject to side load. This load does not cause much compression in the members of the frame, hence the column effect was neglected. In 1961, Ojalvo and Lu[10] gave a method analyzing plane frames of elastic-ideally plastic materials. This method involves the determination of a "compatible" moment and rotation at a joint by the intersection of two moment versus end-rotation curves of the adjoining members. This method has been applied to portal frames and is rigorous. However it may become lengthy for more complicate frames. In 1967, Avarez and Birnstiel[11] showed a method of elastic-plastic analysis of rigid frames of elastic-ideally plastic materials. In 1971, Ridha and Lee[12] presented an interesting theoretical analysis of inelastic two-member plane rigid frames of work-hardening materials by using a variational principle expressed in terms of Kirchoff stress tensor, Green strain tensor and their rates. This method is applicable for frames undergoing moderately large joint rotations and displacements and not specifically for small deflection analysis. The present paper is for small deflection analysis and is simpler than that given by Ridha and Lee. The rigid frame is considered to be subject to such loads that all component members have compressive load small as compared to the critical load, so the beam-column effect is not considered, just like the case given by Ang[19].

METHOD OF ANALYSIS

Slope-deflection method has been used for analyzing elastic rigid frames[13]. *The present study aims to generalize this method for elastic-plastic rigid frames.* A rigid frame is composed of beams and columns. When a beam is loaded beyond the elastic range, the longitudinal stress in some part may exceed the elastic limit. The longitudinal strain e is then composed of the elastic part e^E and the plastic part e^P. The elastic part is related to stress as

$$\sigma = Ee^E = E(e - e^P)$$

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The clasical beam theory is beased on the Bernoulli–Euler assumption that plane sections remain plane during bending. This assumption for bending beyond the elastic range has good experimental corroboration[14, 15] and is used in the present analysis. Consider a beam of uniform cross-section with a plane of symmetry, as shown in Fig. 1. Let x, y and z be a set of rectangular coordinates with x axis along the span of the beam and passing through the centroids of the sections. The load is applied in xz-plane, the plane of symmetry. The displacement along z

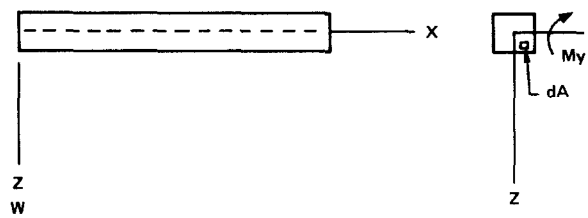

Fig. 1. Sign convention of bending moment of a beam.

axia is denoted by w. Let K be the curvature of the beam. From the Bernoulli–Euler assumption, we have

$$e = e_0 + Kz, \tag{1}$$

the stress

$$\sigma = E(e_0 + Kz - e^P), \tag{2}$$

the axial force

$$F = \int \sigma \, dA = Ee_0A - E\int e^P \, dA = Ee_0A - F^P \tag{3}$$

where

$$F^P = E\int e^P \, dA$$

and the moment

$$M = \int \sigma z \, dA = EIK - E\int e^P z \, dA = EIK - M^P$$

where I is the moment of inertia of the beam, and

$$M^P = E\int e^P z \, dA. \tag{4}$$

For a given beam section of a given stress–strain relation, both M and M^P are functions of the extreme fibre strain[16]. For loading, the moment M is a function of M^P.

$$M(x) = f[M^P(x)] \tag{5}$$

$$\Delta M(x) = f'[M^P(x)]\Delta M^P(x). \tag{6}$$

For unloading ΔM^P vanishes. The curvature and stress are expressed as

$$K = \frac{(M + M^P)}{EI} \cong \frac{d^2 w}{dx^2} \tag{7}$$

$$\sigma = \frac{F + F^P}{A} + \frac{(M + M^P)z}{I} - Ee^P. \tag{8}$$

FIXED END MOMENTS CAUSED BY PLASTIC STRAIN

Consider the beam *AB* in the rigid frame as shown in Fig. 2. After the frame is loaded beyond the elastic range, plastic strain occurs in the frame. At the same time, the joints *A* and *B* move and rotate. The end moments of the beam are considered to be the sum of the fixed end moments due to external load, due to plastic moment M^P in the beam and the end moments caused by rotations and displacements of the joints. To calculate the fixed end moments caused by a known M^P in the beam, the following imaginary processes are used.

Imagine that the beam is cut into many small segments. The left and right sides of each segment are kept plane, but the moment on each segment is relieved. The elastic curvature of the segment is relieved, but the plastic curvature M^P/EI remains. Now *imagine* that a restoring moment $-M^P$ is applied to each segment giving a curvature of $-M^P/EI$ which just cancels M^P/EI. Under such a system of restoring moments, all segments are restored to their original shape and size prior to loading. Hence they match one another perfectly. Imagine that they are welded together. Since the restoring moments of the neighboring segments may be different, there is an unbalanced moment $-\Delta M^P = M_n^P - M_{n+1}^P$ between the nth and $(n+1)$th segments (Fig. 3). This gives an unbalanced moment of $-\partial M^P/\partial x$ per unit length along the span and a bending moment $-M^P$ at the ends. Actually no such restoring moments are applied. These restoring moments are relaxed by applying equal and opposite moments as shown in Fig. 3c. Denoting the moment in the beam caused by this loading (Fig. 3c) as $\bar{M}$, the curvature at any section of the beam is

$$K = \frac{\bar{M}}{EI} \tag{9}$$

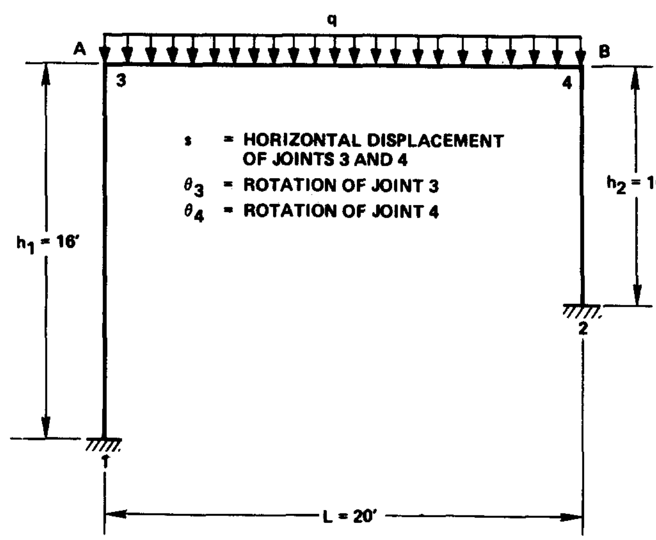

Fig. 2. An unsymmetrical portal frame under loading.

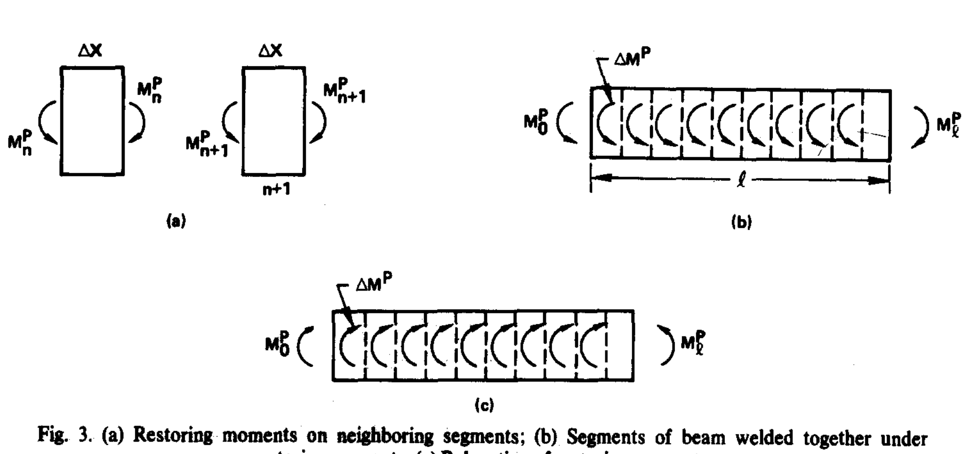

Fig. 3. (a) Restoring moments on neighboring segments; (b) Segments of beam welded together under restoring moments; (c) Relaxation of restoring moments.

The section has $-M^P$ when the curvature is zero, hence with curvature K, the bending moment

$$M = EIK - M^P = \bar{M} - M^P \tag{10}$$

When the loading is applied again, it causes further curvature in the beam. This equation is identical with (7)

$$\bar{M} = M + M^P$$

It is to be noted [16] that this process of cutting, relieving of moment, application of restoring moments, welding, relaxing of the restoring moments and reloading is all *imaginary. There is no further plastic strain developed during this imaginary process.* Under this relaxed moment, which consists of a distributed moment of dM^P/dx and end moments of M_0^P and M_l^P, the fixed-end moments of the beam are calculated as follows.

For a fixed-ended beam subject to a unit moment applied at x', Fig. 4, the end moments are readily calculated as

$$M_1 = \frac{1}{l^2}(l - x')(3x' - l);\ M_2 = \frac{x'}{l^2}(2l - 3x');\ R = -\frac{6x'}{l^3}(l - x') \tag{11}$$

These are influence coefficients for end moments caused by a unit moment applied at x'.

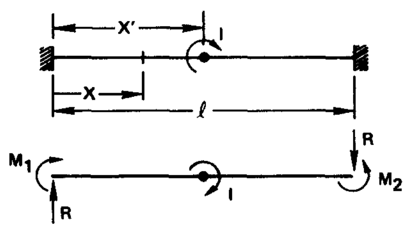

Fig. 4. A beam with fixed ends subject to unit moment at x'.

Under the load of $(\partial M^P(x')/\partial x')$ per unit span M_0^P and M_l^P at the ends, the beam will have end moments

$$\bar{M}_1 = -M_0^P + \int \frac{\partial M^P(x')}{\partial x'} \frac{(l - x')(3x' - l)}{l^2} dx' \tag{12}$$

$$\bar{M}_2 = \int \frac{\partial M^P(x')}{\partial x'} \frac{x'(2l - 3x')}{l^2} dx' + M_l^P \tag{13}$$

Integrating by parts

$$\bar{M}_1 = M_0^P + M^P(x') \frac{(l - x')(3x' - l)}{l^2}\Big|_0^l - \int_0^l M^P(x') \frac{-6x' + 4l}{l^2} dx' = \int_0^l M^P(x')\left(\frac{6x'}{l} - 4\right)\frac{dx'}{l} \tag{14}$$

Similarly

$$\bar{M}_2 = \int_0^l -M^P(x')\left(\frac{6x'}{l} - 2\right)\frac{dx'}{l} \tag{15}$$

Consider in Fig. 5, A to be at the immediate left of $x = 0$ where M_0^P is applied and B at the immediate right of $x = l$ where M_l^P is applied. Hence M_A^P and M_B^P are both zero. From eqn (10), we have

$$\bar{M}_{AB} = M_{AB} = \int_0^l M^P(x')\left(\frac{6x'}{l} - 4\right)\frac{dx'}{l} \tag{16}$$

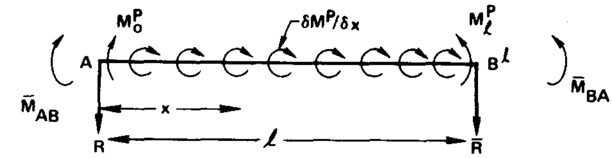

Fig. 5. A free body diagram of the beam.

$$\bar{M}_{BA} = M_{BA} = \int_0^l -M^P(x')\left(\frac{6x'}{l} - 2\right)\frac{dx'}{l} \tag{17}$$

$$R = \frac{\bar{M}_{AB} - \bar{M}_{BA}}{l} = \frac{M_{AB} - M_{BA}}{l} \tag{18}$$

This gives the fixed-end moments and shear force in the beam caused by a given distribution of plastic moment M^P. For numerical calculation, the beam is divided into N-segments. The integrals (16) and (17) may be respectively expressed as

$$M_{AB} = \sum_{n=1}^{N} a_n M_n^{\ P} \tag{19}$$

$$M_{BA} = \sum_{n=1}^{N} b_n M_n^{\ P} \tag{20}$$

These moments will be referred to as plastic fixed-end moments and denoted by $M^P_{F_{AB}}$ and $M^P_{F_{BA}}$.

END MOMENTS

In addition to the fixed end moments due to plastic strains as given above, there are fix-end moments M_{F_1} and M_{F_2} due to external loads. The total fixed end moments are the sums of those due to external loads and due to plastic strain.

In the frame, the joints A and B undergo both angular and linear displacement. The end moment equals the sum of the fixed-end moment, the end moments due to rotation, M_θ and that due to translation, M_Δ

$$M = M_F + M_\theta + M_\Delta.$$

Referring to Fig. 6.

$$M_{1_\theta} + M_{1_\Delta} = \frac{2EI}{l}\left(2\theta_1 + \theta_2 - \frac{3\Delta}{l}\right)$$

$$M_{2_\theta} + M_{2_\Delta} = \frac{2EI}{l}\left(\theta_1 + 2\theta_2 - \frac{3\Delta}{l}\right). \tag{21}$$

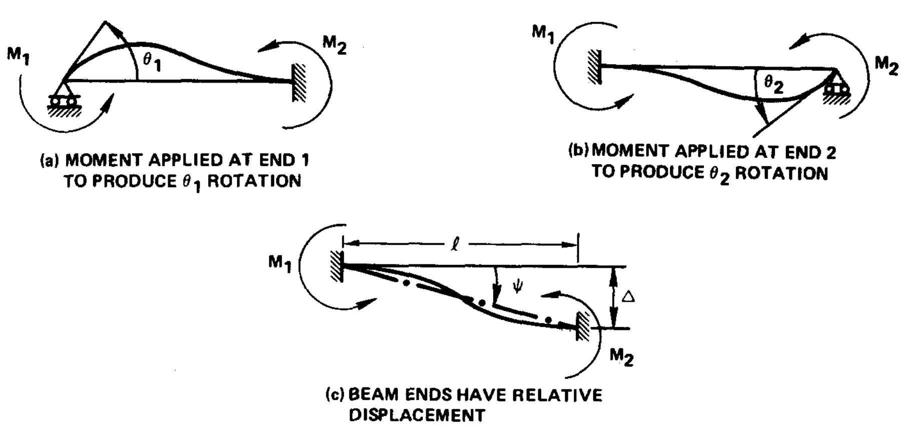

Fig. 6. Fixed end moments caused by rotations and displacements.

Writing the end moment at A of the member AB as M_{AB} and that at B as M_{BA},

$$M_{AB} = \frac{2EI}{l}\left(2\theta_A + \theta_B - \frac{3\Delta}{l}\right) + M_{F_{AB}} + M^P_{F_{AB}} \tag{22}$$

$$M_{BA} = \frac{2EI}{l}\left(2\theta_B + \theta_A - \frac{3\Delta}{l}\right) + M_{F_{BA}} + M^P_{F_{BA}} \tag{23}$$

where the last terms in (22) and (23) are given by (16) and (17) respectively.

Consider an unsymmetrical frame subjected to a uniformly distributed load of intensity q as shown in Fig. 2. The sign conventions for bending moments for frame analysis is shown in Fig. 7. From (22) and (23), we obtain

$$\begin{aligned}
M_{13} &= 2K_{13}\left(\theta_3 - 3\frac{s}{h_1}\right) + M_{F_{13}} + M^P_{F_{13}}\\
M_{31} &= 2x_{13}\left(2\theta_3 - 3\frac{s}{h_1}\right) + M_{F_{34}} + M^P_{F_{34}}\\
M_{34} &= 2K_{34}(2\theta_3 + \theta_4) + M_{F_{34}} + M^P_{F_{34}}\\
M_{43} &= 2K_{34}(\theta_3 - 2\theta_4) + M_{F_{43}} + M^P_{F_{43}}\\
M_{24} &= 2K_{24}\left(\theta_4 - 3\frac{s}{h_2}\right) + M_{F_{24}} + M^P_{F_{24}}\\
M_{42} &= 2K_{24}\left(2\theta_4 - 3\frac{s}{h_2}\right) + M_{F_{42}} + M^P_{F_{42}}
\end{aligned} \tag{24}$$

where K is written for (EI/l) in eqns (22) and (23) and s denotes the sideway of the joints 3 and 4. The equilibrium conditions for joints 3 and 4 give

$$M_{31} + M_{34} = 0, \quad M_{43} + M_{42} = 0 \tag{25}$$

the sum of horizontal shear forces in the left and right columns must vanish.

$$\frac{M_{13} + M_{31}}{h_1} + \frac{M_{24} + M_{42}}{h_2} = 0 \tag{26}$$

Substituting (24) into (25) and (26) yields

$$(4K_{13} + 4K_{34})\theta_3 + 2K_{34}\theta_4 - \frac{6K_{13}}{h_1}s + M_{F_{31}} + M_{F_{34}} + M^P_{F_{31}} + M^P_{F_{34}} = 0 \tag{27}$$

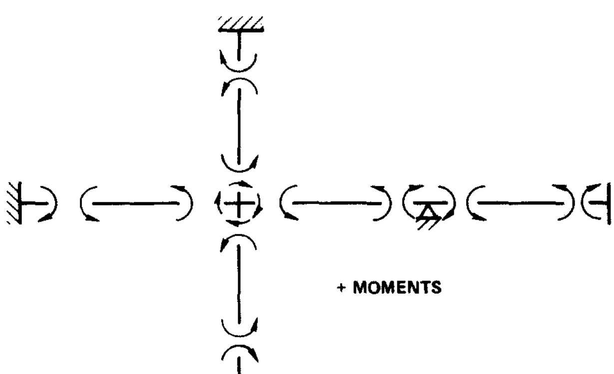

Fig. 7. Sign convention for bending moments in frames.

$$2K_{34}\theta_3+(4K_{34}+4K_{24})\theta_4-\frac{6K_{24}}{h_2}s+M_{F_{43}}+M_{F_{42}}+M^P_{F_{43}}+M^P_{F_{42}}=0 \tag{28}$$

$$6K_{13}\theta_3+6K_{24}\theta_4-\left(\frac{12K_{13}}{h_1}+\frac{12K_{24}}{h_2}\right)s+\frac{M_{F_{13}}+M_{F_{31}}}{h_1}+\frac{M_{F_{24}}+M_{F_{42}}}{h_2}+\frac{M^P_{F_{13}}+M^P_{F_{31}}}{h_1}+\frac{M^P_{F_{24}}+M^P_{F_{42}}}{h_2}=0 \tag{29}$$

For the loading shown in Fig. 2, $M_{F_{13}}=M_{F_{31}}=M_{F_{42}}=M_{F_{24}}=0$

$$M_{F_{34}}=M_{F_{43}}=-\frac{ql^2}{12} \tag{30}$$

Substituting these into (27) to (29) and solving these three simultaneous equations yields

$$\theta=\frac{\begin{vmatrix}\frac{ql^2}{12}-M^P_{F_{31}}-M^P_{F_{34}}, & 2K_{34}, \frac{-6K_{13}}{h_1}\\ \frac{ql^2}{12}-M^P_{F_{43}}-M^P_{F_{42}}, & 4(K_{34}+K_{24}), \frac{-6K_{24}}{h_2}\\ -\frac{M^P_{F_{13}}+M^P_{F_{31}}}{h_1}-\frac{M^P_{F_{24}}+M^P_{F_{42}}}{h_2}, & 6K_{24}, -12\frac{K_{13}}{h_1}+\frac{K_{24}}{h_2}\end{vmatrix}}{D} \tag{31}$$

where

$$D=\begin{vmatrix}4(k_{13}+K_{34}), & 2K_{34} & -\frac{6K_{13}}{h_1}\\ 2K_{34}, & 4(K_{34}+K_{24}) & -\frac{6K_{24}}{h_2}\\ 6K_{13}, & 6K_{24}, & -12\left(\frac{K_{13}}{h_1}+\frac{K_{24}}{h_2}\right)\end{vmatrix} \tag{32}$$

Similarly θ_4 and the sidesway s can be obtained. Substituting these rotations and sidesway into (24), the end moments of the columns and the beam are readily found. Taking each column and beam as a free body, the shear forces at the ends of each member and the bending moment at any section are readily found in terms of "q" and the plastic fixed-end moments (the fixed end moments caused by plastic strain). The shear force and the moment at any section varies linearily with "q" and the plastic end moments, which, in turn vary linearly with plastic moments in all sections in the member. The shear forces at the upper ends of the columns gives the axial forces to the beam and the shear forces of the two ends of the beam give the axial forces to the two columns. Hence the axial forces F and the bending moment at a section in the frame varies linearly with loading "q" and the plastic moments M^P at sections. Let the frame be divided into N-segments. The bending moment M_m and axial force F_m in the mth segment may be expressed as

$$M_m=c_mq+\sum_{n=1}^{N}d_{mn}M_n^P$$

$$F_m=f_mq+\sum_{n=1}^{N}g_{mn}M_n^P \tag{33}$$

where c_m, d_{mn}, f_n and g_{mn} are constants of proportionality. Writing (33) in incremental form

$$\Delta M_m=c_m\Delta q+\sum_{n=1}^{N}d_{mn}\Delta M_n^P \qquad \Delta F_m=f_m\Delta q+\sum_{n=1}^{N}g_{mn}\Delta M_n^P \tag{34}$$

Consider the frame to be of an idealized I-Section as shown in Fig. 8. Each flange area is denoted by A_F and the distance between the two flanges is denoted by h. The stress in the upper flange is considered to be positive for tension and that of the lower flange, positive for compression. The plastic strain e^P vs the stress σ is represented by

$$\begin{aligned} e^P &= f(\sigma) \\ \Delta e^P &= f'(\sigma)\Delta\sigma \quad \text{for} \quad \Delta\sigma \geq 0 \\ &= 0 \qquad\qquad \text{for} \quad \Delta\sigma < 0 \end{aligned} \tag{35}$$

The incremental axial load

$$\begin{aligned} \Delta F &= A_F(\Delta\sigma_2 - \Delta\sigma_1) \\ \Delta M &= A_F\frac{h}{2}(\Delta\sigma_2 + \Delta\sigma_1) \\ \Delta M^P &= EA_F\frac{h}{2}(\Delta e_1{}^P + \Delta e_2{}^P) \\ &= EA_F\frac{h}{2}[f'(\sigma_1)\Delta\sigma_1 + f'(\sigma_2)\Delta\sigma_2] \end{aligned}$$

Eliminating $\Delta\sigma_1$, $\Delta\sigma_2$ from the above three equations, we have

$$\Delta M^P = E\left[f'(\sigma_1)\left(\Delta M - \frac{h}{2}\Delta F\right) + f'(\sigma_2)\left(\Delta M + \frac{h}{2}\Delta F\right)\right] \tag{36}$$

Consider this idealized I-Section to be made of 2024-T3 aluminium alloy with its stress–strain relation represented by

$$e = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^n$$

where $n = 10$, $B = 72.3$ k.s.i. $E = 10^4$ k.s.i. and Poissons' ratio $\partial = 0.33$. The stress–strain relation in compression is assumed to be identical as that in tesion. This gives

$$\begin{aligned} e^P &= \left(\frac{\sigma}{B}\right)^n = f(\sigma) \\ f'(\sigma) &= \frac{n}{B}\left(\frac{\sigma}{B}\right)^{n-1} \end{aligned} \tag{37}$$

Substituting (34) and (37) into (36) yields an equation in terms of ΔM^P's. Writing one such equation for each segment with ΔM^P, we have as many equations as the ΔM^P's. Hence these ΔM^P's in different segments are solved. For each increment of load q, the process is repeated.

For the particular portal frame (Fig. 2), at $q = 8.8$ kips per foot, the moment diagram calculated is shown in Fig. 9 and the flange stresses in the horizontal beam is shown in Fig. 10. This method has also applied to a two-story frame (shown in Fig. 11) made of the same idealized I-Section. The axial and shear forces and end moments for each member is shown in Fig. 12. The load deflection curve for this frame is shown in Fig. 13. This method may be applied to other multistory frames.

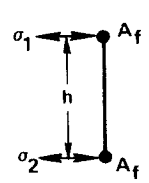

Fig. 8. An idealized I-section.

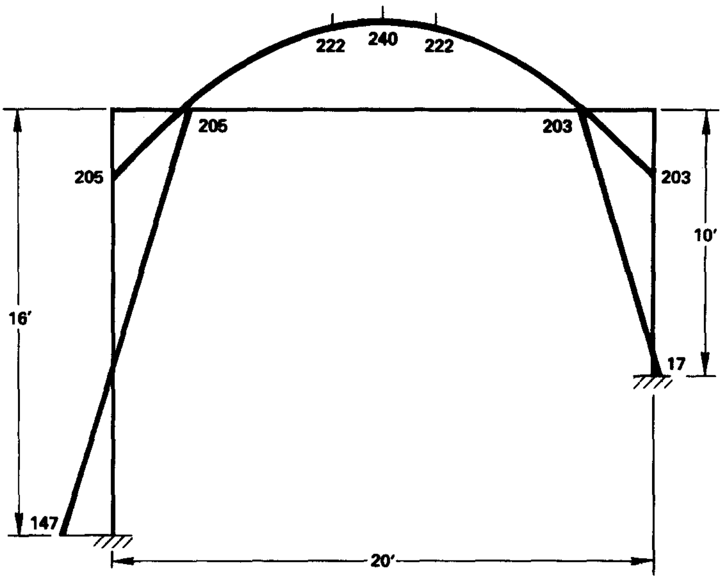

Fig. 9. Bending moment diagram (kip-ft).

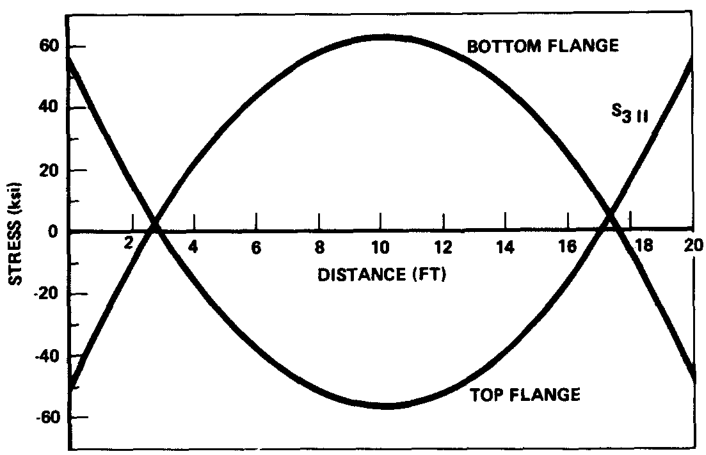

Fig. 10. Stress distribution in the beam.

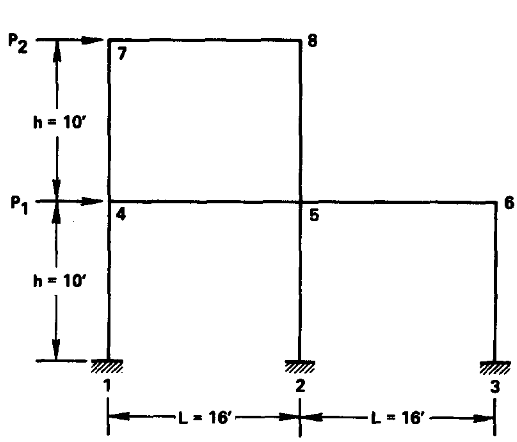

Fig. 11. An unsymmetrical rigid frame under wind load.

CONCLUSIONS

The well-known slope-deflection method for elastic rigid frames is here extended to analyze elastic-plastic frames. The plastic strain is treated as a set of additional applied moments. This method is applicable to frames of work-hardening as well as ideally plastic materials. This method reduces the problem to the solution of a system of simultaneous linear equations. No

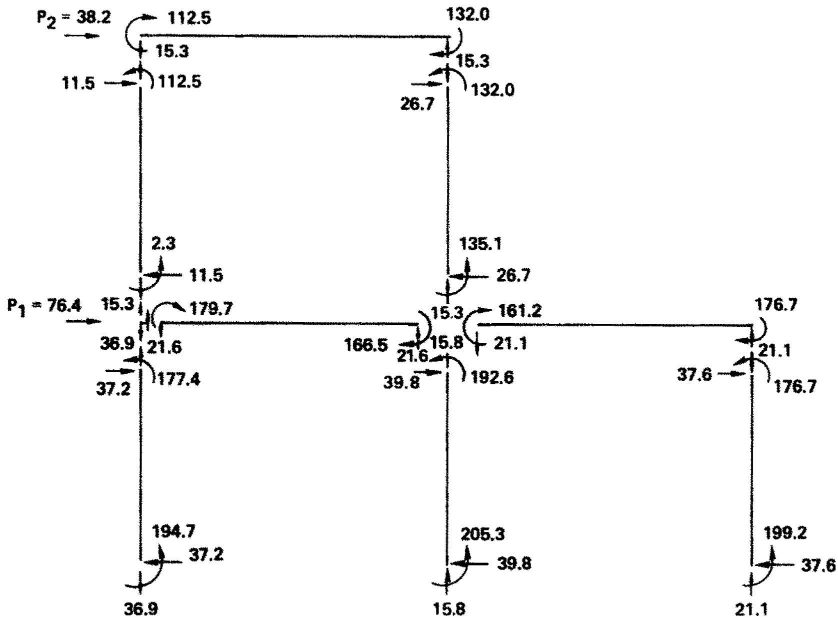

Fig. 12. Axial and shear forces and end moments on each member of the frame. Forces in kips; moments in t-kips.

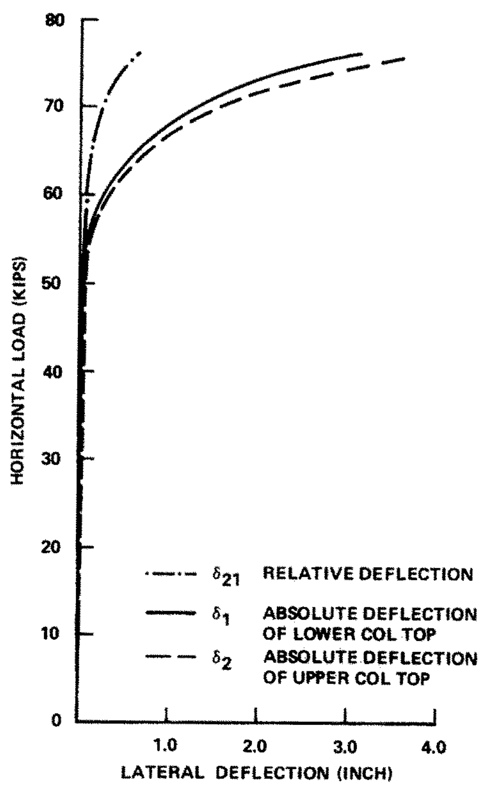

Fig. 13. Lateral deflections of column tops.

iteration is required. In the present analysis, the compressive load in any component member is small as compared to the critical, and the beam-column effect is not considered. Further development of this method is being made to condiser this beam-column effect so as to be applicable to frames with component members subject to elastic-palstic beam-column deformation.

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